The Trouble with Five

Why pentagons refuse to tile a floor — and what Dürer, Kepler and Penrose did about it. A hands-on Space Henge lesson inspired by Plus Magazine's article "The trouble with five."













LEVEL
Grades 6–10 (core); extensions for grades 11–12

TIME
Two 50-minute periods

GROUPS
Groups of 3–4; each needs 1 pentagon pack + 1 decagon pack

NEW PIECES
None. Every build uses the standard kit.

Big idea

Triangles, squares and hexagons tile a flat floor forever. The regular pentagon — the shape of so much in nature, from starfish to flowers — cannot. Its 108° corners never add up to exactly 360°. Mathematicians and artists have wrestled with this "trouble with five" for 500 years: Albrecht Dürer drew pentagon flowers with leftover gaps, Johannes Kepler invented patterns with pentagons, decagons and strange shapes he called "monsters," and Roger Penrose found tilings with five-fold order that never repeat. Students recreate each step with Space Henge's 108° (pentagon) and 144° (decagon) hinges, then discover that pentagons tile perfectly on a **sphere** and on a **saddle** — just not on a flat floor.



108°
n = 5



144°
n = 10

Learning objectives

1. Explain, using angle sums, why regular pentagons cannot tile the plane (3 × 108° = 324° leaves a gap; 4 × 108° = 432° overlaps).
2. Build and analyze patches that combine pentagons and decagons, and identify where and why they break down.
3. Describe five-fold (72°) rotational symmetry and recognize it in the builds.
4. Connect the angle deficit to curvature: pentagons close up into a sphere (dodecahedron) and ruffle into a saddle (hyperbolic plane).
5. Discuss aperiodic tilings (Penrose) and their real-world echo in quasicrystals, and meet the golden ratio in the pentagon.

Standards alignment (Common Core)

- **7.G.B.5 / 8.G.A.5:** Angle facts and informal arguments about angle sums.
- **8.G.A.2–3 / HSG.CO.A.3:** Rotations and symmetry; rotations that carry a figure onto itself.
- **HSG.GMD.B.4:** Relate 2D shapes to 3D objects (the dodecahedron).
- **Mathematical Practices:** MP1 (persevere), MP3 (construct arguments), MP7 (look for structure).

Materials (per group)

- 1 yellow 108° pentagon pack (60 hinges + 30 rods: 12 pentagons)
- 1 white 144° decagon pack (60 hinges + 30 rods: 6 decagons)
- Protractor, ruler, colored paper, scissors, tape
- The student worksheet

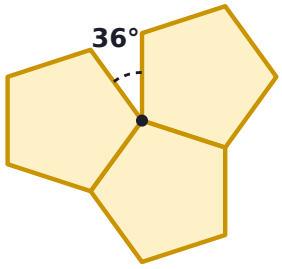
> **Reuse between parts**
> Each part uses at most 12 pentagons and 1 decagon, so groups take apart each build before starting the next. A single Mega Pack (1 pentagon pack, 2 decagon packs) supplies one group; a class of six groups needs six pentagon packs and six decagon packs.

Lesson sequence

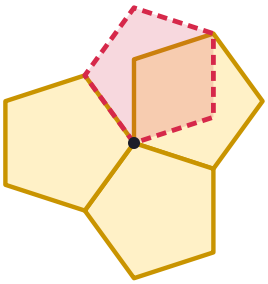
Warm-up: the 360° rule 5 min

Quick review: a flat tiling needs the angles around every corner to add up to exactly 360°. Six 60° triangles, four 90° squares and three 120° hexagons all work. Ask: "What about the 108° pentagon?" Take predictions.

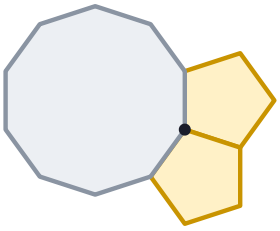
Part 1: the pentagon problem 10 min



3 pentagons: 324°
36° gap



4 pentagons: 432°
72° too much: overlap



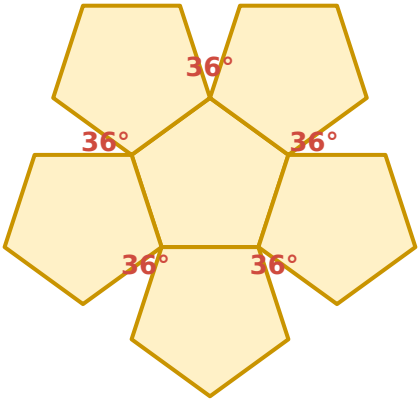
pentagon + pentagon + decagon
exactly 360°

Groups build three pentagons around one point and measure the leftover gap with a protractor (36°). Then they try to squeeze in a fourth: it overlaps by 72°. Conclude: **no number of regular pentagons fits around a point**, so pentagons alone can never tile a floor.

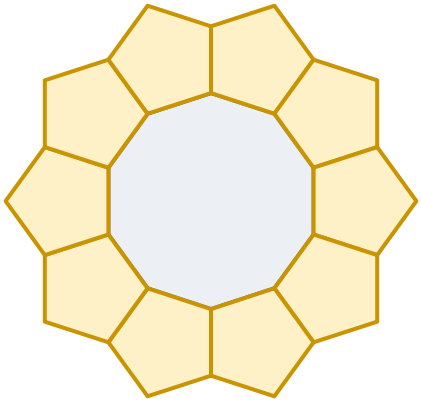
Now try pentagon + pentagon + decagon: 108° + 108° + 144° = **exactly 360°**. The corner closes perfectly. Does that mean pentagons and decagons tile the floor together? Parts 2 and 3 find out.

Part 2: Dürer's flower 15 min

In 1525 the German artist Albrecht Dürer published patterns built from pentagons. Groups build his flower: one central pentagon with a pentagon on each of its five edges (6 pentagons, 30 hinges).



Dürer's flower: 6 pentagons
five 36° notches that no pentagon can fill



Kepler's ring: 1 decagon + 10 pentagons
every inner corner is 108° + 108° + 144° = 360°

- Around each corner of the central pentagon sit three pentagons, so each leaves a **36° notch**. The flower has five of them.
- Turn the build by 72° (one-fifth of a turn). It looks the same: **five-fold symmetry**. Try 90° or 120°: it doesn't.
- Dürer filled such notches with thin diamonds (rhombi) with 36° and 144° corners. Students cut paper rhombi with those angles and tape them in. Note that one corner of the rhombus is 144°, the decagon hinge's angle.

Part 3: Kepler's ring and his "monsters" 15 min

About a century later, in 1619, Johannes Kepler explored pentagon patterns in his book *Harmonices Mundi*. Groups build his ring: one decagon with a pentagon on each of its ten edges (1 decagon + 10 pentagons = 60 hinges).

- Check every inner corner: 108° + 108° + 144° = 360°. The ring is **perfect**, with no gaps at all.
- Now look at the outside edge. Between each pair of neighboring outer pentagons is a 144° notch, exactly the size of a decagon's corner. Challenge: "Keep going by adding decagons into the notches." Students find that neighboring decagons collide. The pattern cannot continue with just these two shapes.
- Kepler's fix was a new shape made by fusing two overlapping decagons, which he called a **"monster."** Every extension of the pentagon idea needs some extra shape. That is the trouble with five.

Part 4: pentagons that *do* work 20 min

The problem is flatness. Change the surface, and pentagons tile beautifully.

- **On a sphere:** three pentagons at a corner fall 36° short, so the surface curves inward. Keep going with the whole pentagon pack: exactly **12 pentagons** close into a **dodecahedron**, a tiling of the sphere. Check: 20 corners × 36° gap = 720°, the total for any closed solid.
- **On a saddle:** four pentagons at a corner have 72° too much, so the surface ruffles like a potato chip. Build a small patch with four pentagons meeting at a point. This is the start of a tiling of the **hyperbolic plane**, where infinitely many regular-pentagon tilings exist.
- Discussion: under 360° curves into a ball, exactly 360° lies flat, over 360° ruffles into a saddle. Pentagons are the one simple shape that works on both curved surfaces but never on a flat one.

Part 5: Penrose, quasicrystals and the golden ratio 15 min, discussion + measurement

- In the 1970s the British mathematician **Roger Penrose** found sets of just two tile shapes that cover the plane with five-fold order but **never repeat**, no matter how far you look. These are aperiodic tilings. (Show a picture of a Penrose tiling. The kit's hinges can't build Penrose's tiles, since they need 36° and 72° corners.)

- In 1982 the chemist **Dan Shechtman** discovered real metal alloys whose atoms are arranged with five-fold order, like a Penrose tiling. Many scientists said it was impossible. He was right and won the **2011 Nobel Prize in Chemistry** for discovering these **quasicrystals**.
- **Golden ratio:** on a built pentagon, measure a diagonal (corner to non-neighboring corner) and a side. Diagonal ÷ side ≈ 1.618, the golden ratio φ. It shows up throughout Penrose tilings and five-fold patterns.

Assessment

Exit ticket.

1. Explain in one or two sentences why regular pentagons can't tile a flat floor.
2. A corner has two pentagons and one decagon. What is the angle sum? Does it lie flat? (108 + 108 + 144 = 360°. Yes.)
3. How many pentagons tile a sphere, and what solid do they make? (12, the dodecahedron.)

Rubric (4 points):

- Uses 324° and 432° (or "gap and overlap") to explain the pentagon problem: 1 point
- Correct 360° check for 5.5.10: 1 point
- 12 pentagons, dodecahedron: 1 point
- Names one way mathematicians worked around the problem (extra shapes, curved surfaces, or aperiodic tilings): 1 point

Differentiation

- **Grades 3–5:** Parts 1, 2 and the dodecahedron only. Focus on "gap," "overlap" and "fits."
- **Support:** give pre-drawn protractor overlays and pre-cut 36°/144° paper rhombi.
- **Challenge:** prove that the diagonal ÷ side of a regular pentagon is $(1 + \sqrt{5}) \div 2$ using similar triangles. Or research the 15 known types of convex (irregular) pentagons that do tile the plane; the last was found in 2015 and the list was proven complete in 2017.

Source and further reading: "The trouble with five," Plus Magazine (plus.maths.org/trouble-five), which covers Dürer's and Kepler's patterns, Penrose tilings, and the still-open five-fold tiling problem in more depth.

Student worksheet: The Trouble with Five

Name _____ Group _____ Date _____

Part 1: around one corner

Shapes around the corner	Angle sum	Gap or overlap?	Lies flat?
3 pentagons			
4 pentagons			
2 pentagons + 1 decagon			

Why can't pentagons alone tile a floor? _____

Part 2: Dürer's flower

Size of each notch: _____° Number of notches: _____ Smallest turn that looks the same: _____°

What shape did Dürer use to fill the notches? _____

Part 3: Kepler's ring

Inner corner check: 108° + 108° + 144° = _____ Size of each outer notch: _____°

What happened when you tried to add decagons to the outside? _____

_____

Part 4: curved surfaces

Pentagons needed to close up: _____ Name of the solid: _____ Gap at each corner: _____°

Describe the shape of the 4-pentagons-per-corner patch: _____

Part 5: golden ratio

Diagonal: _____ cm Side: _____ cm Diagonal ÷ side = _____

Teacher answer key

Part 1.

- 3 pentagons: 324°, 36° gap, doesn't lie flat (curves up).
- 4 pentagons: 432°, 72° overlap, doesn't lie flat (ruffles).
- 2 pentagons + 1 decagon: 360°, exact, lies flat.

Pentagons alone can't tile because 360° isn't a whole-number multiple of 108°.

Part 2. Each notch is 36°, and there are 5. The smallest turn that looks the same is 72° (five-fold symmetry). Dürer filled the notches with thin rhombi with 36° and 144° corners.

Part 3. Inner corners: 360°. Each outer notch is 144° (360° − 108° − 108°). Decagons fit into single notches, but neighboring decagons overlap, so the pattern can't continue without a new shape (Kepler's "monsters").

Part 4. 12 pentagons make a dodecahedron, with a 36° gap at each of its 20 corners (20 × 36° = 720°). The 4-per-corner patch is saddle-shaped: a piece of the hyperbolic plane.

Part 5. Diagonal ÷ side ≈ 1.618 (φ). Allow ±0.05 for measuring error.

Exit ticket.

1. 3 × 108° = 324° leaves a gap; 4 × 108° = 432° overlaps.
2. 360°; yes, it lies flat.
3. 12; the dodecahedron.

Kit contents per spacehenge.xyz: each pack is 60 hinges + 30 rods (octagon packs: 64 + 32). A pentagon pack makes 12 pentagons; a decagon pack makes 6 decagons.