Tile the World

Tessellations with regular polygons: regular and semi-regular tilings, the 360° rule and a famous impossible corner, using the Space Henge kit. A two-period lesson.













LEVEL
Grades 4–9 (core); challenge extensions to grade 12

TIME
Two 50-minute periods

GROUPS
4–6 groups of 3–4 sharing one full kit; floor or big tables

NEW PIECES
None. Every build uses the standard kit.

Big idea

A tessellation covers a flat surface with shapes, with no gaps and no overlaps. The Space Henge hinges are exactly the corner angles of six regular polygons:

- Triangle: 60°
- Square: 90°
- Pentagon: 108°
- Hexagon: 120°
- Octagon: 135°
- Decagon: 144°

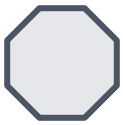
So students can build tilings edge to edge and see why they work. The whole lesson rests on one rule: **the angles that meet at every corner must add up to exactly 360°**. From that rule, students find the only three regular tilings and build six semi-regular tilings. They then meet a corner that adds up to 360° perfectly and still can never tile.


60°
n = 3


90°
n = 4


108°
n = 5


120°
n = 6


135°
n = 8


144°
n = 10

Learning objectives

1. State and use the 360° corner condition for edge-to-edge tilings.
2. Prove that only triangles, squares and hexagons tile the plane on their own.
3. Build semi-regular (Archimedean) tilings and name them with corner notation (such as 4.8.8).
4. Explain why a corner condition that adds up to 360° is *necessary* but not *sufficient* for a tiling, using the 5.5.10 example.
5. Connect angle sums at a corner to curvature: dome, flat and saddle.

Standards alignment (Common Core)

- **4.MD.C.7:** Recognize angle measure as additive; solve for unknown angles.

- **7.G.B.5:** Use facts about supplementary, complementary and adjacent angles.
- **8.G.A.5:** Informal arguments about angle sums.
- **HSG.CO.A.5:** Transformations: translations, rotations and reflections that map a tiling onto itself.
- **Mathematical Practices:** MP1 (persevere), MP3 (construct arguments), MP7 (look for structure).

Materials (per class, from one kit)

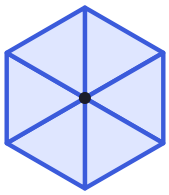
- Hinges: 240 blue 60°, 120 red 90°, 60 yellow 108°, 120 green 120°, 60 gray 135°, 120 white 144°
- Rods: 360 black rods, all the same length, so every tile edge matches
- Painter's tape, sticky notes to mark corners, and the student worksheet

> **How the kit works for tiling**
> In a flat tiling, every polygon corner gets its own hinge, and neighboring tiles **share** the rod between them. So a corner where three tiles meet has three hinges on the rods that end there. Building on the floor keeps big patches flat.

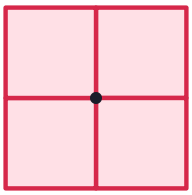
Lesson sequence

Part 1: which single shapes tile? 20 min

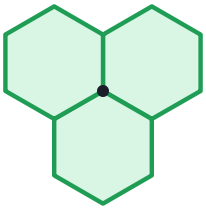
Give each group one hinge color at a time and ask: "Can copies of this one polygon cover the floor around a corner with no gap?" Students fit polygons around a single corner and record the angle sum.



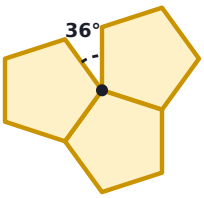
3.3.3.3.3.3
six triangles



4.4.4.4
four squares



6.6.6
three hexagons



5.5.5
three pentagons: 36° gap

Polygon	Angle	360° ÷ angle	Tiles alone?
Triangle	60°	6	Yes: 3.3.3.3.3.3
Square	90°	4	Yes: 4.4.4.4
Pentagon	108°	3.33…	No: three leave a 36° gap and four overlap
Hexagon	120°	3	Yes: 6.6.6 (the honeycomb)
Octagon	135°	2.67	No
Decagon	144°	2.5	No

> **The proof students can own**
> A regular polygon tiles alone only if its angle divides 360° exactly. Once polygons get bigger than hexagons, their angles are above 120°. Then two of them leave a gap, and three already add up to more than 360°. So **exactly three regular tilings exist**, and the kit builds all three.

Build it: each group makes a patch. For example, a 12-triangle patch uses 36 blue hinges, a 3×3 square patch uses 36 red, and a 7-hexagon "flower" uses 42 green.

Part 2: the 360° puzzle 15 min

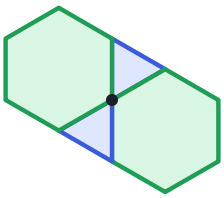
Now mix colors. Challenge: "Find every set of kit polygons whose angles add up to exactly 360° around one corner." Students list the angles, find sums, and check them by building the corner. There are exactly **nine** sets:

- 3.3.3.3.3.3, 4.4.4.4 and 6.6.6 (from Part 1)
- 4.8.8 and 5.5.10
- 3.3.6.6 and 3.4.4.6
- 3.3.3.3.6 and 3.3.3.4.4

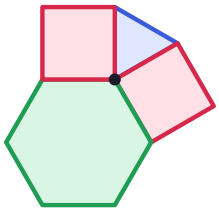
Counting different orders around the corner, three of these sets have two arrangements each: 3.3.6.6 or 3.6.3.6; 3.4.4.6 or 3.4.6.4; and 3.3.3.4.4 or 3.3.4.3.4. That makes 12 corner types in all.

Part 3: build the semi-regular tilings 35 min

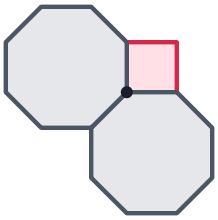
A **semi-regular** (Archimedean) tiling uses two or more regular polygons with **the same arrangement at every corner**. Eight exist, and **six can be built with this kit**. The other two, 3.12.12 and 4.6.12, need a 12-sided polygon's 150° corner. Assign one tiling per group, then do a gallery walk.



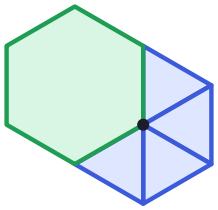
3.6.3.6
trihexagonal



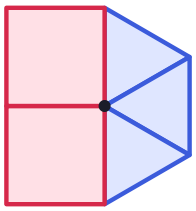
3.4.6.4
rhombitrihexagonal



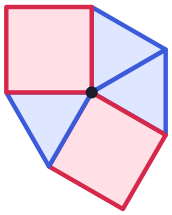
4.8.8
truncated square



3.3.3.3.6
snub hexagonal



3.3.3.4.4
elongated triangular

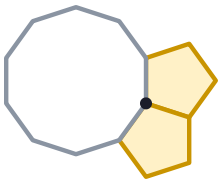


3.3.4.3.4
snub square

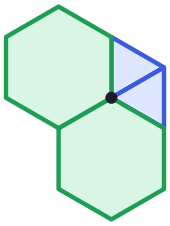
Tiling	Corner	Hinge colors	Look for…
Trihexagonal	3.6.3.6	Blue + green	Straight lines run all the way through: the "Star of David" pattern of basket weaving
Rhombitrihexagonal	3.4.6.4	Blue + red + green	Each hexagon is ringed by squares and triangles
Truncated square	4.8.8	Red + gray	The classic octagon-and-square floor tile
Snub hexagonal	3.3.3.3.6	Blue + green	Comes in left- and right-handed (mirror) versions
Elongated triangular	3.3.3.4.4	Blue + red	Alternating strips of triangles and squares
Snub square	3.3.4.3.4	Blue + red	Squares tilted like a pinwheel

Kit limits for big patches: 60 gray hinges make 7 octagons, so a 4.8.8 patch has about 7 octagons with the squares between them. Green and blue patches can grow much larger.

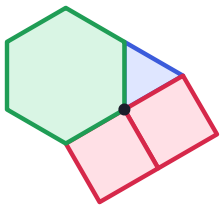
Part 4: the impossible corner 15 min



5.5.10
360° but never tiles



3.3.6.6
only mixed with others

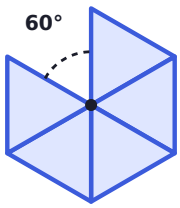


3.4.4.6
only mixed with others

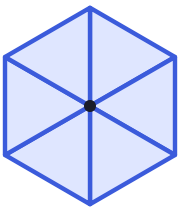
Pentagon + pentagon + decagon adds up to 108° + 108° + 144° = **exactly 360°**. The corner works. Challenge the class to extend it into a tiling. They will fail every time. Why?

> **The odd-cycle argument**
> Every corner must look like 5.5.10. Walk around one pentagon. Across each of its five edges lies either another pentagon or a decagon. At each pentagon corner, the two neighbors across its edges must be one pentagon and one decagon. So going around the pentagon, the neighbors must alternate pentagon, decagon, pentagon, decagon… But five is odd, so the alternation can't close up. **A 360° corner is necessary but not sufficient.** Similarly, 3.3.6.6 and 3.4.4.6 fit around a point but can't repeat everywhere by themselves. They do appear in "2-uniform" tilings, which mix two kinds of corner.

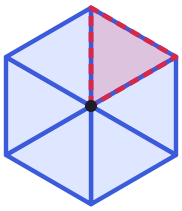
Extension: curvature, from dome to flat to saddle 15 min



5 triangles: 300°
folds into a dome (sphere)



6 triangles: 360°
lies flat (plane)



7 triangles: 420°
60° too many: ruffles into a saddle

What if the angles at a corner *don't* add up to 360°?

- **Less than 360°:** the corner folds up into a dome. Keep going and you get a closed solid. Five triangles at every corner builds the icosahedron, a tiling of the **sphere**.
- **Exactly 360°:** the corner lies flat, and the tiling covers the **plane**.
- **More than 360°:** seven blue triangles at every corner (420°) won't lie flat. The patch ruffles into a wavy, saddle-shaped surface, a model of the **hyperbolic plane**.

Build a seven-triangles-per-corner patch and feel how it grows ruffles. It is the same geometry crocheters use to make coral-reef models.

Extension: why do bees use hexagons?

Build one cell of each regular tiling with the same rods (side s). Compare the area each cell encloses per unit of wall:

- Triangle: about 0.14 s^2 per s of perimeter
- Square: 0.25 s^2
- Hexagon: about 0.43 s^2

Hexagons hold the most honey for the least wax. In fact, no way of dividing the plane into equal areas uses less total wall than hexagons do. Thomas Hales proved this "honeycomb conjecture" in 1999.

Assessment

Exit ticket. Answer all three:

1. Can a regular 12-gon (150°) tile the plane alone? Could it tile with triangles? (No: 360° ÷ 150° = 2.4. Yes: 60° + 150° + 150° = 360°, the 3.12.12 tiling.)
2. Name the tiling with corner 4.8.8 and check its angle sum.
3. Explain in one or two sentences why 5.5.10 fits around a point but can't tile the floor.

Rubric (4 points):

- Both parts of question 1: 1 point
- Correct angle-sum check: 1 point
- The odd-alternation idea: 1 point
- Correct corner notation used throughout: 1 point

Differentiation

- **Grades K–3:** Part 1 only, as a floor challenge: "Which colors make a floor with no holes?"
- **Support:** give a printed corner diagram next to each build station, and pre-counted hinge bags.
- **Challenge:** find the symmetries (translations, rotations and reflections) of each tiling. Or research the two tilings the kit can't build and say which missing hinge they need.
- **Art link:** photograph tilings in school floors, walls and fences. Islamic geometric art and M.C. Escher's prints build on these patterns.

Student worksheet: Tile the World

Name _____ Group _____ Date _____

Part 1: one shape at a time

Polygon	Hinge angle	360° ÷ angle	Does it tile alone? Why?
Triangle			
Square			
Pentagon			
Hexagon			
Octagon			
Decagon			

Rule: a single regular polygon tiles only if _____

Part 2: find every 360° corner

Polygons at the corner (write it like 4.8.8)	Angles added	Total	Did we build it?

Part 3: our semi-regular tiling

Corner notation: _____ Name: _____ Hinges used: _____

Sketch a piece of it:

Part 4: the impossible corner

108° + 108° + 144° = _____ Why can't 5.5.10 cover the floor? _____

_____

Teacher answer key

Part 1.

- Triangle: 60°, 6 fit, tiles.
- Square: 90°, 4 fit, tiles.
- Pentagon: 108°, 3.33, no.
- Hexagon: 120°, 3 fit, tiles.
- Octagon: 135°, 2.67, no.
- Decagon: 144°, 2.5, no.

The rule: the polygon's angle must divide 360° evenly. Only 60°, 90° and 120° do, so there are exactly three regular tilings.

Part 2. The nine sets of kit polygons that add up to 360°:

Set	Angles	Arrangements	Tiles the plane alone?
3.3.3.3.3.3	6 × 60°	1	Yes (regular)
4.4.4.4	4 × 90°	1	Yes (regular)
6.6.6	3 × 120°	1	Yes (regular)
4.8.8	90° + 135° + 135°	1	Yes: truncated square
3.3.3.3.6	4 × 60° + 120°	1	Yes: snub hexagonal (chiral)
{3,3,6,6}	2 × 60° + 2 × 120°	3.6.3.6 / 3.3.6.6	3.6.3.6 yes (trihexagonal); 3.3.6.6 only in mixed tilings
{3,4,4,6}	60° + 2 × 90° + 120°	3.4.6.4 / 3.4.4.6	3.4.6.4 yes (rhombitrihexagonal); 3.4.4.6 only in mixed tilings
{3,3,3,4,4}	3 × 60° + 2 × 90°	3.3.3.4.4 / 3.3.4.3.4	Both yes: elongated triangular and snub square
5.5.10	2 × 108° + 144°	1	Never (odd-cycle argument)

So the kit builds all three regular tilings and six of the eight semi-regular ones: 9 of the 11 uniform tilings. The missing two, 3.12.12 and 4.6.12, need a 150° dodecagon hinge.

Part 4. 360°. Around any pentagon, the neighbors across its edges would have to alternate between pentagon and decagon, and five edges is an odd number, so the pattern can't close.

Exit ticket.

1. No alone (2.4 doesn't divide evenly); yes with triangles, as 3.12.12.
2. The truncated square tiling: 90° + 135° + 135° = 360°.
3. See Part 4.

Kit contents per spacehenge.xyz: 720 hinges (240 × 60°, 120 × 90°, 60 × 108°, 120 × 120°, 60 × 135°, 120 × 144°) and 360 black rods, all one length. Each angle is also sold as a 60-hinge pack with 30 rods.