Angles You Can Hold

Polygon angles, the five Platonic solids, and Euler and Descartes with the Space Henge kit. A two-to-three-period geometry lesson.













LEVEL
Grades 6–10 (core); challenge extensions for grades 11–12

TIME
Two to three 50-minute periods

GROUPS
4–6 groups of 3–4 sharing one full kit

NEW PIECES
None. Every build uses the standard kit.

Big idea

Every Space Henge hinge is locked to one angle: 60°, 90°, 108°, 120°, 135° or 144°. These are not random. They are exactly the interior angles of the regular triangle, square, pentagon, hexagon, octagon and decagon. Students start by discovering the formula that produces these numbers. They then ask what happens when corners are folded up off the table. That question leads straight to one of the oldest results in mathematics: there are exactly five Platonic solids. Along the way it reaches two beautiful counting laws, Euler's V − E + F = 2 and Descartes' "720° rule."



60°
n = 3



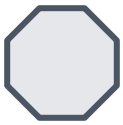
90°
n = 4



108°
n = 5



120°
n = 6



135°
n = 8



144°
n = 10

Learning objectives

1. Derive and apply the interior-angle formula for a regular polygon, (n − 2) · 180° ÷ n, and the angle-sum formula, (n − 2) · 180°.
2. Show that the exterior angles of any convex polygon add up to 360°.
3. Distinguish *equilateral* from *regular* polygons.
4. Explain why exactly five Platonic solids exist, using the angle sum at a corner.
5. Verify Euler's formula (V − E + F = 2) and Descartes' angle-defect theorem (total defect = 720°) on built solids.

Standards alignment (Common Core)

- **7.G.B.5:** Use facts about supplementary and adjacent angles to find unknown angles (exterior angles).
- **8.G.A.5:** Use informal arguments to establish facts about angle sums and exterior angles.
- **HSG.GMD.B.4:** Identify three-dimensional objects and relate them to two-dimensional shapes.
- **HSG.MG.A.1:** Use geometric shapes and their properties to describe objects.
- **Mathematical Practices:** MP3 (construct arguments), MP5 (use tools strategically), MP7 (look for structure).

Materials (per class, from one kit)

- Hinges: 240 blue 60°, 120 red 90°, 60 yellow 108°, 120 green 120°, 60 gray 135°, 120 white 144°
- Rods: 360 black rods, all the same length, so every edge in every build matches
- Protractors, painter's tape and the student worksheet

> **How the kit works**
> A hinge sits at one corner of one face and joins that corner's two rods. Hinges rotate on rods, so flat polygons fold along shared rods into 3D solids. Two counting rules follow for any closed build: **hinges = 2 × rods**, and **hinges = total number of face corners**.

Lesson sequence

Part 1: where do the hinge angles come from? 20 min

Each group builds one regular polygon of each kind using a single hinge color and equal rods: 3 blue, 4 red, 5 yellow, 6 green, 8 gray and 10 white. For each one, students record the number of sides *n*, the hinge angle, and the angle sum (*n* × hinge angle).

n	Polygon	Hinge angle	Angle sum	(n − 2) · 180°	Exterior angle (180° − hinge)	n × exterior
3	Triangle	60°	180°	180°	120°	360°
4	Square	90°	360°	360°	90°	360°
5	Pentagon	108°	540°	540°	72°	360°
6	Hexagon	120°	720°	720°	60°	360°
8	Octagon	135°	1080°	1080°	45°	360°
10	Decagon	144°	1440°	1440°	36°	360°

- **Pattern hunt:** each extra side adds 180°. Why? Pick one corner and draw diagonals from it: an n-gon splits into (n − 2) triangles.
- **Formula:** for a regular n-gon, each angle is (n − 2) · 180° ÷ n. Have students check it against every hinge.
- **Why no 7-gon hinge?** The formula gives 900° ÷ 7 ≈ 128.57°, which is not a whole number. A 9-gon needs 140°. Ask: which n give whole-number angles? (Those where n divides 360: 3, 4, 5, 6, 8, 9, 10, 12 and so on.)

Part 2: the 360° walk 10 min

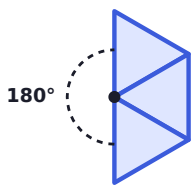
Lay a big hexagon on the floor. One student walks the perimeter, turning at each corner, and the class counts the degrees turned. Each turn is the **exterior angle**, 180° minus the hinge angle. Repeat with the triangle and the decagon. The total is always 360°: one full turn, whatever the shape. That's the last column of the table, and it gives a second route to the interior-angle formula.

Part 3: equilateral is not the same as regular 10 min

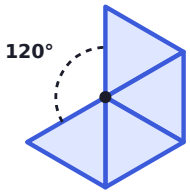
- Every Space Henge rod is the same length, so every shape you build is **equilateral** (all sides equal).
- Build a four-sided shape from two blue 60° hinges and two green 120° hinges, alternating. All four sides are equal, but the angles aren't. It is a rhombus: equilateral but not regular.
- Challenge: "Build a rectangle that isn't a square." Students discover it's impossible with this kit, because a rectangle needs two different side lengths. A regular polygon needs **both** equal sides and equal angles; this kit guarantees the first, and the hinges choose the second.

Part 4: why exactly five Platonic solids? 30 min

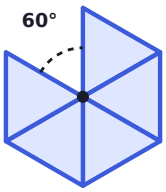
Fit regular polygons around a single corner and look at the gap left over. If the angles add up to **less than 360°**, the gap can be folded closed and the corner pops up into 3D. If they add up to **exactly 360°**, the corner lies flat. If they add up to **more**, the polygons don't fit at all.



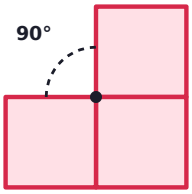
Tetrahedron
3 triangles · gap 180°



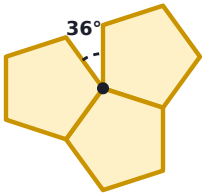
Octahedron
4 triangles · gap 120°



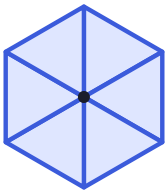
Icosahedron
5 triangles · gap 60°



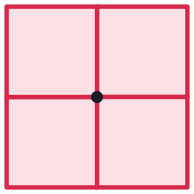
Cube
3 squares · gap 90°



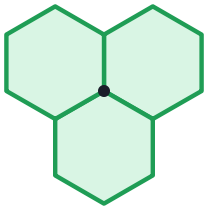
Dodecahedron
3 pentagons · gap 36°



Six triangles
no gap: lies flat



Four squares
no gap: lies flat



Three hexagons
no gap: lies flat

A corner needs at least three faces, and every face must be the same regular polygon, with the same arrangement at every corner. With those rules, only five corners work:

- 3, 4 or 5 triangles (180°, 240° or 300°)
- 3 squares (270°)
- 3 pentagons (324°)

Six triangles, four squares and three hexagons each reach 360° and lie flat. Anything bigger overlaps. So there are exactly five Platonic solids, and the kit builds all five:

Solid	Faces	Hinges	Rods (E)	Corners (V)	Faces (F)	V − E + F
Tetrahedron	4 triangles	12 blue	6	4	4	2
Cube	6 squares	24 red	12	8	6	2
Octahedron	8 triangles	24 blue	12	6	8	2
Dodecahedron	12 pentagons	60 yellow	30	20	12	2
Icosahedron	20 triangles	60 blue	30	12	20	2

Part 5: Descartes' 720° rule 15 min

At each corner of a solid, the **angle defect** is 360° minus the angles that meet there. That is the gap from Part 4. Add up the defect over every corner:

Solid	Defect per corner	× corners	Total
Tetrahedron	180°	× 4	720°
Cube	90°	× 8	720°
Octahedron	120°	× 6	720°
Dodecahedron	36°	× 20	720°
Icosahedron	60°	× 12	720°

The surprise

Every closed convex solid, however it's built, has a total angle defect of exactly 720°, two full turns. This is Descartes' theorem, and it is equivalent to Euler's V − E + F = 2. A shape with small gaps at each corner needs more corners to add up to 720°. That's why the dodecahedron (36° gaps) needs 20 corners while the tetrahedron (180° gaps) needs only 4.

Capstone: the Archimedean challenge

one period or a take-home project

Now allow **more than one kind of regular polygon**, still with the same arrangement at every corner. These are the 13 Archimedean solids, and the kit's hinge counts can build every one of them, one at a time. The largest, the truncated icosidodecahedron, uses exactly all 120 red, 120 green and 120 white hinges. Groups pick a solid, predict its hinges and rods from the corner arrangement, then build it and verify V − E + F = 2 and the 720° total defect.

Archimedean solid	Corner (faces around it)	Hinges needed	Rods	V	Defect per corner
Truncated tetrahedron	3.6.6	12 blue + 24 green	18	12	60°
Cuboctahedron	3.4.3.4	24 blue + 24 red	24	12	60°
Truncated cube	3.8.8	24 blue + 48 gray	36	24	30°
Truncated octahedron	4.6.6	24 red + 48 green	36	24	30°
Rhombicuboctahedron	3.4.4.4	24 blue + 72 red	48	24	30°
Snub cube	3.3.3.3.4	96 blue + 24 red	60	24	30°
Truncated cuboctahedron	4.6.8	48 red + 48 green + 48 gray	72	48	15°
Icosidodecahedron	3.5.3.5	60 blue + 60 yellow	60	30	24°
Truncated dodecahedron	3.10.10	60 blue + 120 white	90	60	12°
Truncated icosahedron (soccer ball)	5.6.6	60 yellow + 120 green	90	60	12°
Rhombicosidodecahedron	3.4.5.4	60 blue + 120 red + 60 yellow	120	60	12°
Snub dodecahedron	3.3.3.3.5	240 blue + 60 yellow	150	60	12°
Truncated icosidodecahedron	4.6.10	120 red + 120 green + 120 white	180	120	6°

Corner notation lists the polygons around one corner by number of sides: 3.4.3.4 means triangle, square, triangle, square. Every row's defect × V = 720°. Prisms (two n-gons joined by squares) and antiprisms (two n-gons joined by triangles) for n = 3, 4, 5, 6, 8 and 10 make easy warm-up builds.

Assessment

Exit ticket. Answer all three:

1. What is each interior angle of a regular 12-gon? Is there a Space Henge hinge for it? (Answer: 150°. No.)
2. Why can't three regular hexagons form the corner of a solid?
3. A solid has 30 corners, each with a 24° defect. Check it against Descartes' rule. (30 × 24° = 720°. ✓)

Rubric (4 points):

- Correct 150°: 1 point
- Hexagon explanation that uses the 360° sum: 1 point
- Descartes check: 1 point
- Clear written reasoning: 1 point

Differentiation

- **Grades 3–5:** just Parts 1 and 2, plus building a cube and a tetrahedron. Count faces, edges and corners.
- **Support:** give pre-counted bags of hinges for each build, and a partially filled table.
- **Challenge:** prove that $V - E + F = 2$ follows from the 720° rule. (Hint: the angle sum of all faces is $\Sigma(n_i - 2) \cdot 180° = (2E - 2F) \cdot 180°$.) Or ask why a snub cube comes in left- and right-handed versions.

Student worksheet: Angles You Can Hold

Name _____ Group _____ Date _____

Part 1: regular polygons

n	Hinge color	Hinge angle	Angle sum (n × angle)	Exterior angle	n × exterior
3					
4					
5					
6					
8					
10					

Angle sum formula: _____ Each angle of a regular n-gon: _____

Why is there no 7-gon hinge? _____

Part 4: corners

Polygons around one corner	Angle sum	Gap (360° − sum)	Folds up, lies flat, or doesn't fit?
3 triangles			
4 triangles			
5 triangles			
6 triangles			
3 squares			
4 squares			
3 pentagons			
3 hexagons			

How many Platonic solids are there, and why? _____

Parts 4 and 5: solids we built

Solid	Hinges	Rods (E)	Corners (V)	Faces (F)	V − E + F	Defect per corner	Total defect

Teacher answer key

Part 1. Angle sums: 180°, 360°, 540°, 720°, 1080°, 1440°. Sum = (n − 2) · 180°, and each angle = (n − 2) · 180° ÷ n. Exterior angles: 120°, 90°, 72°, 60°, 45°, 36°, and n × exterior = 360° every time. A 7-gon needs 128.57…°, which is not a whole number, and 360 is not divisible by 7.

Part 4. Angle sums and gaps:

- 3 triangles: 180°, gap 180°, folds up (tetrahedron)
- 4 triangles: 240°, gap 120°, folds up (octahedron)
- 5 triangles: 300°, gap 60°, folds up (icosahedron)
- 6 triangles: 360°, no gap, lies flat
- 3 squares: 270°, gap 90°, folds up (cube)
- 4 squares: 360°, no gap, lies flat
- 3 pentagons: 324°, gap 36°, folds up (dodecahedron)
- 3 hexagons: 360°, no gap, lies flat

Four pentagons (432°) and anything with larger polygons overlap. So there are exactly five.

Part 5. See the Platonic table: V − E + F = 2 for all five, and total defect = 720° for all five.

Exit ticket.

1. 150°; no hinge exists for it.
2. 3 × 120° = 360° leaves no gap to fold, so the corner lies flat.
3. 30 × 24° = 720°, which matches the icosidodecahedron.

Challenge proof. Total defect = $360°\cdot V - \Sigma(\text{face angle sums}) = 360°\cdot V - \Sigma(n_i - 2)\cdot 180° = 360°\cdot V - (2E - 2F)\cdot 180° = 360°\cdot(V - E + F)$. So a total of 720° means V − E + F = 2.

Kit contents per spacehenge.xyz: 720 hinges (240 × 60°, 120 × 90°, 60 × 108°, 120 × 120°, 60 × 135°, 120 × 144°) and 360 black rods, all one length. Each angle is also sold as a 60-hinge pack with 30 rods.